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Tightening Penrose Inequality

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Penrose inequality:

The surface area of the Kerr horizon is

$$A_h = 8\pi M_{\mathrm{B}}\left(M_{\mathrm{B}} + \sqrt{M_{\mathrm{B}}^2 - a^2}\right) \leq 16\pi M_{\mathrm{B}}^2$$

Following these three points:

1. Since the gravitational waves carry positive energy, the Bondi mass can never be larger than the Arnowitt-Deser-Misner (ADM) mass M

2. From the second law of black hole thermodynamics, the area of the event horizon can not decrease

3. Any apparent horizon must be hidden inside the event horizon

Penrose inequality is an inequality on initial data

$$\boxed{2M \geq \sqrt{\frac{A[\sigma]}{4\pi}}}$$

The Penrose inequality involves the apparent horizon of the initial data set, and it does not require that the spacetime is or will finally settle into a stationary black hole. It gives a relation between the energy and the size of the space it occupies. If we focus on a stationary black hole, the area of horizon stands for the Bekenstein-Hawking entropy of the system via

$$S = \frac{1}{4}A$$

The Penrose inequality then becomes an entropy bound for a system of given total energy.

The fact that the product TS has the same dimension of energy suggests that it may modify the mass bound in the inequality

$$\boxed{3M \geq \sqrt{\frac{S}{\pi}} + 2TS}$$

This inequality is independent of the Penrose inequality and the saturation appears to be possible also only by the Schwarzschild black hole.

An interesting question arises: could we find a new lower bound for the total energy such that it is tighter than these inequalities?

Penrose inequality is about how the appropriate energy-momentum tensor of the minimally-coupled matter constrains the spacetime geometry.

We observe from the RN black hole that $TS = \frac{r_+^2 - 4Q^2}{4r_+}$ $M = \frac{4Q^2 + r_+^2}{2r_+}$ $\longrightarrow$ $M = r_+ - 2TS$

We propose a new Penrose-like inequality for a static black hole $\boxed{M \geq \sqrt{\frac{S}{\pi}} - 2TS}$

For the extremal case, we have T = 0 and the above inequality gives us a tighter bound of the ADM mass compared to the Penrose inequality

The Proof:

For the spherically-symmetric and static configurations

$$ds^2 = -f(r)e^{-\chi(r)}dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2$$

The asymptotic flatness requires $f(r) = 1 - 2M/r + \cdots\,, \qquad \lim_{r\to\infty} r\,\chi(r) = 0$

For the matter energy momentum tensor $T^{\mu}{}_{\nu} = \mathrm{diag}\{-\rho(r), p_r(r), p_T(r), p_T(r)\}$ Einstein's equation leads to three ordinary differential equations

$$f' = \frac{1 - 8\pi r^2\rho - f(r)}{r} \qquad \chi' = \frac{-8\pi r(\rho + p_r)}{f} \qquad p_r' = \frac{\rho - 3p_r + 4p_T}{2r} - \frac{(\rho + p_r)(1 + 8\pi p_r r^2)}{2fr}$$

There exists four known quasi-local masses associated with (Monotonically non decreasing function)

NEC $\rho + P \geq 0$ $\quad m_n(r) = \frac{r^4 e^{\chi/2}}{6}\left(\frac{fe^{-\chi}}{r^2}\right)' + \frac{r}{3}$ $\quad m_n(r_+) = \frac{2TS}{3} + \frac{r_+}{3}$ $\quad m_n(\infty) = M$

WEC $\rho \geq 0$ & $\rho + P \geq 0$ $\quad m_w(r) = \frac{r}{2}(1 - f)$ $\quad m_w(r_+) = \frac{1}{2}r_+$ $\quad m_w(\infty) = M$

SEC $\rho + 3P \geq 0$ & $\rho + P \geq 0$ $\quad m_s(r) = \frac{1}{2}r^2 e^{\chi/2}(fe^{-\chi})'$ $\quad m_s(H) = \frac{1}{2}TS$ $\quad m_s(\mathcal{S}_\infty) = M$

DEC $\rho \geq |P|$ $\quad m_d(r) = \frac{r}{2}(1 - e^{-\chi/2}f)$ $\quad$ Give us same as SEC and WEC

We define a quasi-local mass that is the most general linear combination of all other masses

$$m(r) = 2m_w(r) - m_s(r) + \alpha(3m_n(r) - m_s(r) - 2m_w(r)) + 2\beta(m_d - m_w)$$

such that

$$m(r_+) = r_+ - 2TS \qquad m(\infty) = M$$

It's derivative yields

$$m'(r) = \gamma(1 - e^{-\frac{1}{2}\chi}) + (1-\gamma)\pi r^2 \rho(1 - e^{-\frac{1}{2}\chi}) + 4\pi r^2 e^{-\frac{1}{2}\chi}\Big((1-\gamma)\rho - (1+\gamma)p_r - 2p_T\Big)$$

Imposing NEC+TEC (−T ≥ 0) to the above relation we have $m'(r) \geq 0$

Note that the WEC will be also satisfied under requirement of NEC+TEC.

Rotating case

$$M + 2TS \geq \sqrt{\frac{S}{\pi} - |J|}$$

$\frac{S}{\pi} \geq 2|J|$ KN and Kerr-Sen

$$M + 2TS \geq \sqrt{\frac{S}{\pi} - |J|} \geq |J|$$

D dimensional case

$$M + \frac{D-2}{D-3}TS \geq \frac{D-2}{\pi}\Big(\frac{\Omega_{D-2}}{2^D}\Big)^{\frac{1}{D-2}} S^{\frac{D-3}{D-2}}$$

$$\Omega_k = 2\pi^{\frac{k+1}{2}}/(\tfrac{1}{2}(k-1))$$

Schwarzschild-Tangherlini

RN-Tangherlini

Cosmological constant case

$$M + 2TS - 4\,P_{\rm th}V_{\rm th} \geq \sqrt{\frac{S}{\pi}}$$

Schwarzschild-(A)dS and RN-(A)dS

Einstein-Born-Infeld-(A)dS

Einstein-QTE-(A)dS

Summary of the concert examples:

BH/EC	**NEC**	**WEC**	**DEC**	**SEC**	**NEC+TEC**	inequality
RN	True	True	True	True	True	True
Born-Infeld	True	True	True	True	True	True
QTE ($\alpha \geq 0$)	True	True	True	True	True	True
STU	True	True	True	True	True	True
Pure scalar	True	False	False	False	False	True
Bardeen	True	True	False	False	False	False

Conclusion :

Penrose inequality for extremal black holes have a large gap from saturation, so we proposed a tighter bound for static black holes which is saturated by the RN black hole.

We gave a proof for general spherically-symmetric and static black holes.

The requirement “NEC+TEC” is a sufficient condition for our inequality.

We considered generalizations in various directions such as to higher dimensions, to include rotation and cosmological constant.

Tightening the Penrose Inequality

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ABSTRACT

ne Penrose inequality estimates the lower bound of the mass of a black hole in

rea of its horizon. This bound is not very “tight” for extremal or near extren

Thank you for your attention